

Vertex Operators in $2K$ Dimensions

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A formula is proposed which expresses free fermion fields in $2K$ dimensions in terms of the Cartan currents of the free fermion current algebra. This leads, in an obvious manner, to a vertex operator construction of nonabelian free fermion current algebras in arbitrary even dimension. It is conjectured that these ideas may generalize to a wide class of conformal field theories.

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1. Introduction

The vertex operator construction of two dimensional current algebras, and the associated procedure of bosonization of fermions[1] are powerful tools for obtaining nonperturbative information about two dimensional field theory. The purpose of the present note is to derive an analogous formula in any number of dimensions, in particular 4. We will present precise results only for theories of free fermions, although in the penultimate section we will also speculate about current algebras in nontrivial conformal field theories.

The basic feature of the vertex operator construction is that it enables us to construct charged operators out of the diagonal currents of the algebra, which are themselves uncharged under the Cartan subalgebra. A hint that a similar construction may be possible in other dimensions is Schwinger's proof that an abelian spatial current density always carries charge density in a positive metric quantum field theory. Formally the Schwinger charge density integrates to zero, but one might imagine that a sufficiently nonlocal functional of the current might actually carry charge. The challenge is to find such a functional which is also a local operator.

The real clue to generalizing the bosonization formulae to higher dimensions comes from a convenient rewriting of the two dimensional formulae. Recall that a Weyl fermion can be constructed as

$$\psi(z) = e^{-i\phi(z)} \quad (1.1)$$

where the holomorphic current operator $j(z)$ is

$$j(z) = i\partial_z\phi(z) \quad (1.2)$$

We rewrite this as

$$\psi(z) = e^{i \int A_\mu(w, \bar{w}) j^\mu(w, \bar{w}) d^2 w} \quad (1.3)$$

where

$$A_w = \frac{1}{(w - z)} \quad (1.4)$$

$$A_{\bar{w}} = \frac{1}{(\bar{w} - \bar{z})} \quad (1.5)$$

Notice that this vector potential has delta function field strength concentrated at z . It is a singular instanton.

The fundamental reason that bosonization works then is that the Ward identity for a charged field

$$\partial_{\bar{z}} j(z) \psi(w) = \delta^2(z - w) \psi(w) \quad (1.6)$$

is just the anomaly equation

$$\partial_{\bar{z}} j(z) = \epsilon^{\mu\nu} F_{\mu\nu}(z) \quad (1.7)$$

for the chiral current in an instanton background. In order to generalize this observation to multiple field correlators, one uses the abelian nature of the background gauge field.

Another aspect of this two dimensional observation that will be of use in higher dimensions is the following. What we have said is that studying the insertion of a single fermion operator at z is the same as studying the theory in the background of a singular instanton located at z . Consider multifermion correlators in such a background. $U(1)$ charge conservation plus the fact that the fermions are free, tell us that the only nonvanishing connected correlation function is the one point function of the conjugate fermion field $\bar{\psi}$.

To obtain the fermion two point function, we consider the *partition function* of the system in the presence of a singular instanton-antiinstanton pair located at 0 and z . This is the determinant of the Weyl operator and requires careful definition. We define the real part of the log of the determinant as one half the Schwinger result for a full Dirac fermion (with the gauge field coupled to the vector current). The imaginary part is defined by the anomaly equation. The singular instanton is almost everywhere a pure gauge, and the variation of the determinant with respect to the gauge function is $\int i\delta\theta \rho_{top}$, where ρ_{top} is the topological charge density. Plugging in the singular gauge function and topological charge density of an instanton anti-instanton pair, we determine the phase of the determinant. In this calculation, infinite, z independent, constants in the log of the determinant are dropped and the normalization of the two point function is thus determined by hand. This procedure gives the correct result $\frac{1}{z}$. Multipoint correlators are obtained from multiple instanton pairs. They produce the correct free fermion behavior as a consequence of

the bilinearity of Schwinger's formula for the fermion determinant. The generalization of this procedure to $2K$ dimensions will be given below.

2. $2K$ Dimensional Vertex Operators

The generalization of what we have just said to four (and indeed to any even number of dimensions) is moderately straightforward. We work in Euclidean metric and place a single Weyl fermion field in a singular instanton field with delta function topological charge, normalized so that it creates a single unit of chiral charge at the point x . Such an instanton can be constructed out of two dimensional instantons as follows. Choose a set of K mutually orthogonal planes in $2K$ dimensions. We will denote a generic point in the space of all such ordered sequences of mutually orthogonal planes by P , and planes belonging to the set by $i \in P$. Choose unit vectors \mathbf{n}_i in each plane. On each plane, place a two dimensional instanton with delta function field strength on that plane. If $\omega_{\mu\nu}^i$, $i = 1 \dots K$, are the area tensors of the planes (i.e. constant antisymmetric tensors equal to the two dimensional $\epsilon_{\mu\nu}$ when the indices are in the appropriate plane, and zero otherwise), and

$$P_{\mu\nu}^i = \sum_{a=1}^2 \delta_{\mu a} \delta_{\nu a}$$

(with a the two orthogonal directions in the plane i) the projection onto the plane, then the full gauge configuration is

$$A_\mu = \sum_i A_\mu^i = \sum_i \frac{\omega_{\mu\nu}^i x^\nu}{x^T P^i x} \quad (2.1)$$

This configuration has a unit topological charge located at $x = 0$. Another way to write this configuration is to note that A_μ is a singular pure gauge, $\partial_\mu \sum \theta_i(x)$, where the θ_i are the angles between the projection of x onto the plane i and the unit vector \mathbf{n}_i .

A vertex operator will be an expression of the form $e^{i \int A_\mu J^\mu}$, where J^μ is the chiral fermion number current. The fact that it depends on a choice of planes shows that it will not be a scalar under rotation. To understand how it transforms we first note that there is in fact a whole multiplet of vertex operators with the same topological charge even for a given choice of planes. Indeed we can change the sign of an even number of

the planar instantons, without changing the topological charge. The number of different vertex operators with a given orientation of the planes is thus 2^{K-1} . which is precisely the number of components of a Weyl spinor in $2K$ dimensions. This motivates the following construction:

The Weyl matrices in $2K$ dimensions are $2^{K-1} \times 2^{K-1}$ dimensional matrices combined together into the $2K$ vector $\sigma^\mu = (i, \gamma^a)$ where γ^a is a hermitian representation of the $2K - 1$ dimensional Dirac algebra. They satisfy the algebra

$$\sigma^\mu \sigma^{\dagger \nu} = \delta^{\mu\nu} + i \Sigma^{\mu\nu} \quad (2.2)$$

$\Sigma^{\mu\nu}/2$ are the generators of the Weyl representation of $O(2K)$. These matrices are hermitian and satisfy $(\Sigma^{\mu\nu})^2 = 1$ for each plane. The Cartan subalgebra consists of the Σ matrices for K mutually orthogonal planes. Note that one of these can, without loss of generality be chosen to be one of the $(2K - 1)$ dimensional Dirac matrices while the others are commutators of independent pairs of the $2K - 2$ remaining Dirac matrices. Odd dimensional Dirac matrices satisfy the identity

$$\prod \gamma^a = 1 \quad (2.3)$$

so only $(K - 1)$ of the eigenvalues of the Cartan generators can be chosen independently.

Now let φ_α^n be Weyl spinors which are simultaneous eigenvectors of the Cartan matrices. Introduce a set of fermionic annihilation operators C_n . Define a vertex operator by

$$V_\alpha = (e^{i \int \sum_i A_\mu^i \Sigma^i J^\mu})_\alpha^\beta \sum_n c_n \varphi_\beta^n \quad (2.4)$$

Here Σ^i are the $\Sigma^{\mu\nu}$ corresponding to the set of orthogonal planes we have chosen for the instanton. As φ runs over the independent eigenvectors of the Cartan generators, these vertex operators run over the 2^{K-1} possible choices of sign described above.

To discuss the transformation properties of the vertex operator, let us imagine doing an $O(2K)$ transformation on the fundamental fermion fields in the functional integral. The current transforms like a $2K$ dimensional vector field, so by changing variables in the space time integral in the exponent of the vertex operator, we see that the effect of this

operation is to rotate the vector potential. A general rotation will rotate the planes P into another collection, ΛP , of mutually orthogonal planes. The vertex operator has become

$$V'_\alpha = (e^{i \int_{i_R} A_\mu^{i_R \Sigma^i} J^\mu})_\alpha^\beta \varphi_\beta, \quad (2.5)$$

where i_R is the plane into which i is rotated. Now let $\mathcal{D}(R_{i_R})$ be the representation matrix in the Weyl spinor representation of the rotation which takes the collection of planes $[i]$ into the collection $[i_R]$. Then

$$V'_\alpha = \mathcal{D}(R_{i_R})_{\alpha\gamma} (e^{i \int_i A_\mu^{\Lambda P \Sigma^{i_R}} J^\mu})_\gamma^\delta \mathcal{D}^\dagger(R_{i_R})_{\delta\beta} \varphi_\beta \quad (2.6)$$

Now note that the matrix \mathcal{D}^\dagger , operating on the eigenstates φ of Σ^i , produces eigenstates of Σ^{i_R} with the same eigenvalue.

$$\mathcal{D}^\dagger \varphi_{(P)}^n = e^{i\delta_n(\Lambda)} \varphi_{(\Lambda P)}^n \quad (2.7)$$

In order to cancel the phases in this transformation we must allow the c_n to transform under Euclidean rotations. They are thus additional dynamical variables in our system. The operators c_n are the analogs of the zero mode fermion introduced in the sixth reference of [1]. In two dimensions, bilinears in these zero mode fermions produce the cocycle operators of the Frenkel-Kac construction.

With these rules, it is then clear that the vertex operator transforms under an Euclidean rotation as a Weyl spinor:

$$V_\alpha(x) \rightarrow \mathcal{D}_\alpha^\beta(\Lambda) V_\beta(\Lambda^{-1}x) \quad (2.8)$$

2.1. Green Functions

In order to compute fermion correlation functions in terms of determinants in background multiinstanton fields, we use the same prescription that worked in two dimensions. If the gauge field is coupled to a Dirac fermion in a vectorlike manner, there are two fermion zero modes for the Dirac operator in a field with topological charge one, and the Abelian analog of the 't Hooft interaction is bilinear in the fermions. The determinant in

an instanton anti-instanton background is thus proportional to $(x_I - x_{\bar{I}})^{-(4K-2)}$, *i.e.* it scales like the square of the fermion propagator¹. Note that this calculation is valid for any choice of the eigenvalues of the Σ_i . This means that if we define the absolute value of the Weyl determinant to be the square root of the Dirac determinant, then it is proportional to the unit matrix in Weyl spinor space. The same will not be true of the phase of the Weyl determinant. The full determinant will be a matrix $D_{\dot{\beta}\dot{\delta}}^{\alpha\gamma}(x_I - x_{\bar{I}})$ and the vertex operator two point function will be

$$\langle V_\alpha(x_I) V_{\dot{\beta}}(x_{\bar{I}}) \rangle \propto D_{\dot{\beta}\dot{\delta}}^{\alpha\gamma}(x_I - x_{\bar{I}}) \langle c_n \bar{c}_m \rangle \varphi_n^\gamma \bar{\varphi}_m^{\dot{\delta}} \quad (2.9)$$

Here \bar{c}_m are the zero mode operators for the antifermion field.

In order to complete the calculation, we must specify the two point function of the zero mode operators. Define $\mathbf{N} \equiv \frac{1}{\sqrt{K}} \sum \mathbf{n}_i$. We insist that

$$\langle c_n \bar{c}_m \rangle \varphi_n^\gamma \bar{\varphi}_m^{\dot{\delta}} = (N_\mu \sigma^{\dagger\mu})_{\gamma\dot{\delta}} \quad (2.10)$$

Note that in four dimensions any matrix connecting the spaces of the two spinor representations is a linear combination of the σ^\dagger , so we are only requiring that $\sqrt{K}\mathbf{N}$ have unit normalized components in two orthogonal planes. In higher dimensions, we could in principle have obtained a more complicated matrix.

Now let us return to the phase of the Weyl determinant. The anomaly equation determines it to be

$$\frac{1}{2} \int [\sum (\theta_i(x - x_I) \Sigma^i + \theta_i(x - x_{\bar{I}}) \bar{\Sigma}^i) \rho_{top}] \quad (2.11)$$

(The matrices in this equation operate in the tensor product of the two spinor representations. Thus, the unbarred Σ does not operate on the barred indices and vice versa.). The topological charge density is

$$\rho_{top} = \delta^{2K}(x - x_I) - \delta^{2K}(x - x_{\bar{I}}) \quad (2.12)$$

¹ Generally, this scaling behavior is just the long distance limit of the determinant. However, neither the singular instantons nor the theory contain a scale, so apart from renormalizations of the vertex operators, the long distance limit is the whole function.

The phase (2.11) thus contains some undefined pieces proportional to $\theta_i(0)$ which we absorb into the normalization of the vertex operators. The finite part comes from the cross terms and has the form

$$\frac{1}{2} \sum (\theta_i(x_I - x_{\bar{I}}) \Sigma^i + \theta_i(x_I - x_{\bar{I}}) \bar{\Sigma}^i) \quad (2.13)$$

This is just the Weyl representative of the rotation that rotates the vector $\Delta = \mathbf{x}_I - \mathbf{x}_{\bar{I}}$ into the direction of the vector \mathbf{N} . Acting on $N_\mu \sigma^{\dagger\mu}$ it produces $\hat{\Delta}_\mu \sigma^{\dagger\mu}$. Combining this with the square root of the Dirac determinant, we obtain precisely the free fermion propagator.

3. Conformally Invariant Speculations

In two dimensions, the vertex operator construction is the basis of a remarkable simplification of any conformal field theory with continuous global symmetries. The ability to build operators of arbitrary transformation properties out of the currents themselves, suggests that the currents completely decouple from the rest of the theory. By multiplying an arbitrary charged operator by an appropriate function of the currents one can construct a bleached operator which commutes with the currents. The bleached operators are generally nonlocal, but there are local functions of them which also commute with the currents. These form a separate conformal field theory with their own stress tensor. To see this one constructs the affine-Sugawara stress tensor from the currents, and verifies that it is conserved, traceless and satisfies the Virasoro algebra. The coset stress tensor obtained by subtracting $T_{Sugawara}$ from the full stress tensor satisfies the Virasoro algebra with a complementary value of the central charge. It is the stress tensor of the bleached variables. The Hilbert space of the theory is a tensor product of charged and bleached sectors.

How much of this can we expect to generalize to higher dimensions? One result that certainly does not generalize is the formula for an abelian current in terms of a free boson field. In higher dimensions, the currents of free fermions have nonvanishing connected n point functions. Indeed the anomaly is just the connected 3 point function. However, it is not inconceivable that the rest of the two dimensional results generalize. If we take any subset of free fermion currents which forms a closed operator product algebra, then the operator product expansion (OPE) of pairs of currents contains a dimension four conserved

symmetric traceless tensor $T_{\mu\nu}$ which is a partial stress tensor and generates the correct equations of motion for the currents. Perhaps this is a special feature of the free fermion theory but one may conjecture that it generalizes to interacting field theories.

There is an obvious check of this conjecture which I have not yet carried out. Gauge theories with large numbers of colors and flavors have perturbatively accessible fixed points. This was first pointed out in [2]². One can check the OPE of two currents at these perturbative but nontrivial fixed points, to see whether the dimension of the symmetric traceless tensor operator in this OPE changes. If it remains equal to four, then the stress tensor at the nontrivial fixed point will decompose into two commuting pieces.

In this case we would conjecture that a four dimensional conformal field theory containing fermion fields would decompose into a tensor product of a “conformal current algebra” theory and a bleached theory. Note the restriction to theories containing fermions. A crucial part of the argument is our ability to build charged fields out of neutral ones. As we have seen, this is a consequence of the anomalous violation of fermion current conservation in the presence of instantons. I know of no analogous phenomenon for purely bosonic canonical variables in four dimensions.

4. Applications?

The higher dimensional bosonization formulae that we have constructed are certainly more intricate than their two dimensional counterparts. How do they compare in utility? Certainly the formula hints at a more fundamental understanding of what fermions really are, but it remains to be seen whether it can lead to solutions of models that cannot be understood in any other manner.

The most promising avenue of research in this regard, and one of the primary motivations for the present work, is the profound generalization of electric magnetic duality discovered by Seiberg in supersymmetric nonabelian gauge theories[3]. At the core of

² Actually, in this reference the perturbative fixed point was accessed by taking the number of flavors to be fractional. This fractional flavor theory is well defined, but not unitary. I believe that it was D.Gross who first pointed out that for large N one could achieve a perturbative fixed point with integer number of colors and flavors, thus preserving unitarity.

Seiberg's argument for duality is a conformal field theory that appears to be infinitely strongly coupled in terms of the original (electric) variables.

Our bosonization formulae can be generalized to multifermion theories that contain nonabelian global symmetries which we can gauge. As in two dimensions, the theory at finite gauge coupling looks ugly when written in terms of bosonized variables (the cartan subalgebra of gauge currents)³, and is not conformally invariant. However, the infinitely strongly coupled theory is formally obtained by modding out the free theory by the gauge group. That is, locally gauge invariant operators which do not contain the gauge fields retain their free field form, and gauge invariance is simply the statement that the nongauge invariant operators do not act on the physical Hilbert space.

By analogy with orbifold theories in two dimensions, one may then expect to find that there exist operators in the free theory which are not local with respect to the fundamental fermion fields, but are local with respect to all gauge invariant functions of them. I conjecture that these may be the “magnetic quarks” of Seiberg's duality transformation.

I have not yet been able to construct such magnetic operators. Furthermore, it is clear that there is more to the story than the outline above. Seiberg's construction leads us to expect free magnetic gauge bosons at the conformal point. How are these to be constructed? In addition, his dual theory contains elementary fields corresponding to certain gauge invariant fermion bilinears. These have different dimension at the free magnetic fixed point than they do at the free electric fixed point, contradicting the orbifold analogy propounded above. It is perhaps superfluous to state that more research along these lines is necessary .

³ We have constructed the higher dimensional analog of abelian but not of nonabelian bosonization.

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